ON THE SOLUTION OF PLANE IRROTATIONAL HYDRODYNAMIC PROBLEMS FOR DOUBLY-CONNECTED REGIONS

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PMM Vol.25, No.1, 1961, pp. 145-147

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(Received June 16, 1960)

The case of plane flow about a single body, when the exterior or the interior of a circle maps conformally onto the exterior of the contour of the body, is considered in detail in the literature and does not offer any special mathematical difficulties. It is, however, more complicated to solve problems of plane flow in a doubly-connected region; for conditions where a known function conformally maps an annulus Σ, with radii ρ_1 and ρ_2 onto the doubly-connected region S which the flow occupies (we assume that the region S contains the point at infinity), the solution may be unwieldy. We shall show here that, as soon as one knows the function

$$z = \omega(\zeta) \qquad (z = x + iy,\ \zeta = \rho e^{i\vartheta}) \tag{1}$$

which gives the conformal mapping of the annulus onto the exterior of two contours, the velocity potential ϕ can be formed immediately.

We assume that the contour C_1 of the region S corresponds to the circumference of radius $\rho = 1$ in the region Σ. The ratio of the radii $\rho_1/\rho_2 = 1/\rho_2$ is determined geometrically by the form of the region S.

In (1), separating the real and imaginary parts, we get $x = x(\rho,\ \theta)$, $y = y(\rho,\ \theta)$ where ρ and θ are curvilinear coordinates in the region S. The velocity potential satisfies Laplace's equation, which in curvilinear coordinates has the form

$$\nabla^2\varphi = \frac{\partial}{\partial\rho}\left(\frac{H_\vartheta}{H_\rho}\ \frac{\partial\varphi}{\partial\rho}\right) + \frac{\partial}{\partial\vartheta}\left(\frac{H_\rho}{H_\vartheta}\ \frac{\partial\varphi}{\partial\vartheta}\right) = 0 \tag{2}$$

Here H_ρ and H_θ are the Lamé parameters for the ρ- and θ-coordinates (where ρ means the coordinate normal to the curves ρ = const in the direction of increasing ρ)

$$H_\rho^2 = \left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2, \qquad H_\vartheta^2 = \left(\frac{\partial x}{\partial \vartheta}\right)^2 + \left(\frac{\partial y}{\partial \vartheta}\right)^2 \tag{3}$$

Because the mapping (1) is conformal, $x(\rho, \theta)$ and $y(\rho, \theta)$ are connected by the Cauchy-Riemann conditions

$$\frac{\partial x}{\partial \rho} = \frac{1}{\rho}\frac{\partial y}{\partial \vartheta}, \qquad \frac{\partial y}{\partial \rho} = -\frac{1}{\rho}\frac{\partial x}{\partial \vartheta} \tag{4}$$

On the basis of (3) and (4) we have $H_\rho/H_\theta = 1/\rho$, and Equation (2) takes the form

$$\frac{\partial^2 \varphi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \varphi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \varphi}{\partial \vartheta^2} = 0 \tag{5}$$

Furthermore, it will be shown that the boundary conditions for the problem considered, the plane irrotational flow about two cylindrical bodies (or the motion of two bodies in an unbounded fluid) have much the same form as in the Neumann problem for the annulus. Moreover, in the expansion of $\partial \phi / \partial \rho$ on the contours into a trigonometric series the free term is not present and so the solution of Equation (5) is sought in the form [5]

$$\varphi = B_0\vartheta + \sum_{m=1}^{\infty} (A_m\rho^m + A_{-m}\rho^{-m})\cos m\vartheta + \sum_{m=1}^{\infty} (B_m\rho^m + B_{-m}\rho^{-m})\sin m\vartheta \tag{6}$$

We shall solve the problem of determining the velocity potential in a curvilinear coordinate net. The net will be taken so that the contours of the cylinders C_1 and C_2 are defined by the curvilinear coordinates $\rho = 1$ and $\rho = \rho_2$.

1. We shall first consider the case of the plane-parallel motion of two bodies in an unbounded flow. We shall limit ourselves to translational motion with equal velocity vectors **V**. In the usual case of motion the ratio $\rho_1/\rho_2 = 1/\rho_2$ will be some function of time t. The velocity potential ϕ satisfies Equation (5) and the following boundary conditions:

On the contours of the cylinders we have the condition of non-penetration

$$\frac{\partial \varphi}{\partial n} = \frac{1}{H_\rho}\frac{\partial \varphi}{\partial \rho} = V_x \cos(\rho, x) + V_y \cos(\rho, y) \tag{7}$$

with

$$\cos(\rho, x) = \frac{1}{H_\rho}\frac{\partial x}{\partial \rho}, \qquad \cos(\rho, y) = \frac{1}{H_\rho}\frac{\partial y}{\partial \rho} \qquad \left(\frac{1}{H_\rho} \neq 0, \quad H_\rho = |\omega'(\zeta)|\right) \tag{8}$$

then substituting (8) into (7), we have for $\rho = \rho_1 = 1$

$$\frac{\partial\varphi}{\partial\rho} = V_x \frac{\partial y}{\partial\vartheta} - V_y \frac{\partial x}{\partial\vartheta} = \mu_1(\vartheta) = \sum_{m=1}^{\infty} (\alpha_m^{(1)} \cos m\vartheta + \beta_m^{(1)} \sin m\vartheta) \qquad (9)$$

For $\rho = \rho_2$

$$\frac{\partial\varphi}{\partial\rho} = \frac{1}{\rho_2}\left(V_x \frac{\partial y}{\partial\vartheta} - V_y \frac{\partial x}{\partial\vartheta}\right) = \mu_2(\vartheta) = \sum_{m=1}^{\infty} (\alpha_m^{(2)} \cos m\vartheta + \beta_m^{(2)} \sin m\vartheta) \qquad (10)$$

$\mu_1(\theta)$ and $\mu_2(\theta)$ can easily be found if the mapping function (1) is given.

The constants in the series (9) and (10) are absent; it is not difficult to convince oneself of this by expanding $\omega(\zeta) = x(\rho, \theta) + iy(\rho, \theta)$ into a Fourier series for $\rho = 1$ and $\rho = \rho_2$.

Substituting (6) into (9) and (10) and equating coefficients of the same trigonometric functions, we obtain two systems of two equations with two unknowns, defining A_m, A_{-m}, B_m, B_{-m}.

For the velocity potential we obtain

$$\varphi = B_0\vartheta + \sum_{m=1}^{\infty} \frac{(\alpha_m^{(1)}\rho_2^{-m-1} - \alpha_m^{(2)})\rho^m + (\alpha_m^{(1)}\rho_2^{m-1} - \alpha_m^{(2)})\rho^{-m}}{m(\rho_2^{-m-1} - \rho_2^{m-1})}\cos m\vartheta +$$
$$+ \sum_{m=1}^{\infty} \frac{(\beta_m^{(1)}\rho_2^{-m-1} - \beta_m^{(2)})\rho^m + (\beta_m^{(1)}\rho_2^{m-1} - \beta_m^{(2)})\rho^{-m}}{m(\rho_2^{-m-1} - \rho_2^{m-1})}\sin m\vartheta = B_0\vartheta + \varphi_0(\rho, \vartheta) \qquad (11)$$

It is evident that finding $\phi_0(\rho, \theta)$ reduces to the solution of a Neumann problem for an annulus [2]. The function $\phi_0(\rho, \theta)$ represented by series (11) is harmonic in the ring $(1, \rho_2)$ and it follows that it is unique and continuous together with its derivatives up to and including the second order. The constant B_0 is determined by the given circulation on one of the contours; for $\rho = 1$

$$\Gamma = \int_s \frac{\partial\varphi}{\partial s} ds = \int_0^{2\pi} \frac{\partial\varphi}{\partial\vartheta} d\vartheta = 2\pi B_0 \quad \text{or} \quad B_0 = \frac{\Gamma}{2\pi} \qquad (12)$$

For the solution of the problem of the motion of two bodies in an unbounded fluid the velocity corresponding to the chosen potential ϕ must be zero at infinity. We shall check whether this condition is fulfilled. Let the point λ of the region correspond to the point at infinity of the region S. Because the mapping (11) is one to one then the function $\zeta = F(z)$ accomplishes the conformal mapping of the doubly-connected region S onto the circular ring Σ. It is known that the isolated singular points $\zeta = \lambda$ (poles) of the function $\omega(\zeta)$ determine the point of intersection $(\partial F/\partial z = 0)$ of the inverse function [4], so that for $|z| \to \infty$

$$\frac{d\zeta}{dz} = \frac{1}{\omega'(\zeta)} = 0 \quad \text{or} \quad \frac{\partial\rho}{\partial x} = \frac{\partial\rho}{\partial y} = \frac{\partial\vartheta}{\partial x} = \frac{\partial\vartheta}{\partial y} = 0 \quad (\text{for } \zeta = \rho e^{i\theta}) \qquad (13)$$

We now show that the velocity corresponding to the potential ϕ determined from Formula (13) actually equals zero at infinity, i.e. for $|z| \to \infty$

$$V_x^{\infty} = \frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial \varphi}{\partial \vartheta}\frac{\partial \vartheta}{\partial x} = 0, \quad V_y^{\infty} = \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial \varphi}{\partial \vartheta}\frac{\partial \vartheta}{\partial y} = 0 \tag{14}$$

Because $\phi(\rho, \theta)$ is a harmonic function in the ring, $\partial \phi / \partial \rho$, $\partial \phi / \partial \theta$ take on finite values at all points of the ring (and it follows also in the region S) including the point $\zeta = \lambda$, and on the basis of (13) it is not difficult to see the correctness of (14).

2. We solve analogously the problem of the potential flow about two cylinders. We take the flow such that the x- and y-components of velocity at infinity are V_x^{∞}, V_y^{∞}.

We shall seek ϕ in the form

$$\varphi = V_x^{\infty} x + V_y^{\infty} y + \varphi_1 \tag{15}$$

where ϕ_1 satisfies Equation (5) and can be found in the same form as (6). The constants

$$A_m,\ A_{-m},\ B_m,\ B_{-m}$$

are determined from the boundary conditions. On the contour of the cylinders

$$\frac{\partial \varphi}{\partial n} = 0, \quad \text{or} \quad \frac{1}{H_\rho}\frac{\partial \varphi}{\partial \rho} = 0, \quad \text{for} \quad \rho = 1 \text{ and } \rho = \rho_2 \tag{16}$$

Inserting (15) into (16) we get

$$\frac{\partial \varphi_1}{\partial \rho} = -V_x^{\infty}\frac{\partial y}{\partial \vartheta} + V_y^{\infty}\frac{\partial x}{\partial \vartheta} \quad \text{for } \rho = 1 \tag{17}$$

$$\frac{\partial \varphi_1}{\partial \rho} = \frac{1}{\rho_2}\left(-V_x^{\infty}\frac{\partial y}{\partial \vartheta} + V_y^{\infty}\frac{\partial x}{\partial \vartheta}\right) \quad \text{for } \rho = \rho_2 \tag{18}$$

Comparing (17) and (18) with (9) and (10) we conclude that the finding of ϕ_1 is identical with the previous problem, except that V_x and V_y have to be changed into $-V_x^{\infty}$ and $-V_y^{\infty}$. As in the previous case the velocities corresponding to ϕ_1 equal zero at infinity. Knowing $\phi(\rho, \theta)$ it is not difficult to find the projection of the velocity vector in the directions ρ and ϕ at any desired point

$$V_\rho = \frac{1}{H_\rho}\frac{\partial \varphi}{\partial \rho} = \frac{1}{|\omega'(\zeta)|}\frac{\partial \varphi}{\partial \rho}, \quad V_\vartheta = \frac{1}{H_\vartheta}\frac{\partial \varphi}{\partial \vartheta} = \frac{1}{\rho\,|\omega'(\zeta)|}\frac{\partial \varphi}{\partial \vartheta} \tag{19}$$

From (19) it is seen that the problem is solved in the curvilinear coordinates obtained by the conformal mapping of a circular ring onto the given region.

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Translated by H.C.